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Random Linear Programs with Many
Variables and Few Constraints

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Abstract

We extend and simplify Smale's work on the expected number of
pivots for a linear program with many variables and few constraints.
Our method applies to new versions of the simplex algorithm and new
random distributions.

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and Few Constraints

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1. Introduction. In the important papers [1,2], Smale studies

the expected number of pivots required by the simplex algorithm for a

randomly generated problem.

Smale considers a version of the simplex algorithm which can be

viewed as a path-following procedure. For a fixed constraint matrix

one begins with a right-hand-side and objective function for which

the optimal solution is trivial and deforms to the actual ones. The

"main formula" in [1,2] gives the expected number of pivots for a fixed

matrix. The "main formula" is then used to prove the "main theorem"--

that for a fixed number of constraints the expected number of steps

grows in a sublinear manner as the number of variables increases.

In this note we show how the "main theorem" can be obtained without

using the geometric analysis leading up to the "main formula." This

simplifies the proof. At the same time the result is generalized to

include other versions of the simplex algorithm and other random dis-

tributions of the data, which are not necessarily amenable to geometric

analysis.

2. Statement of the Problem. We will consider linear programs in

the form

$$\max cx$$
$$Ax \geq b \qquad\qquad\qquad (2.1)$$
$$x \geq 0 \qquad\qquad [b \in R^m, \ c \in R^n]$$

We will assume b is a fixed non-positive* vector and study the expected number of pivots when A and c are generated randomly. The case in which b also varies can be obtained as a corollary. The crucial property assumed of the random distribution is

Let $A' = \binom{c}{A}$. For any (m+1) by n matrix B with all elements different

$$\text{prob}\{a'_{ij} > a'_{ik} \text{ if } b_{ij} > b_{ik} \text{ for all } i,j,k\} = (\frac{1}{n!})^{m+1}$$

(2.2)

In words (2.2) says that the order of elements in c and the rows of A are independent of one another. This is clearly the case if A and c are chosen using the spherical measure of [1], but includes other possibilities. (2.2) does imply that the probability that any two members of the same row of A are equal is zero, and is essentially equivalent to the symmetry assumption in [2]. In section 5 we show how our analysis can be extended to those distributions satisfying

Let B and B* be two matrices such that each row of one is a permutation of the corresponding row of the other. Then

(2.2)'

$$\text{prob}\{a'_{ij} > a'_{ik} \text{ iff } b_{ij} > b_{ik}; \text{ all } i,j,k\} =$$
$$\text{prob}\{a'_{ij} > a'_{ik} \text{ iff } b^*_{ij} > b^*_{ik}; \text{ all } i,j,k\}$$

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*To insure feasibility. This assumption can be avoided by minor modifications, which we indicate in section 5.

(2.2)' would allow, for example, choice of all elements of A and c from independent identically distributed discrete random variables.

3. <u>Idea of the proof.</u> Consider the LP

$$\max 10x_1 + 5x_2 + 9x_3 + 4x_4$$
$$-x_1 - 2x_2 - 7x_3 - 10x_4 \geq -5 \qquad\qquad (3.1)$$
$$-7x_1 - 8x_2 + 3x_3 + x_4 \geq -15$$
$$x_1, \ x_2, \ x_3, \ x_4 \geq 0$$

We can see immediately that the optimal solution to (3.1) must have $x_2 = x_4 = 0$. This is because the column of A' $(10,-1,-7) \geq$ $(5,-2,-8)$ and $(9,-7,3) \geq (4,-10,1)$. In general a column of A' is said to be undominated if there is no other column at least as large in every row.* We are concerned with those versions of the simplex algorithm which satisfy

No variable corresponding to a dominated column of A'

enters the basis at any iteration.

$$(3.2)$$

The path-following algorithm in [1,2] satisfies (3.2). Other versions of the simplex algorithm satisfying (3.2) include

If possible, choose a surplus variable as entering

variable. Otherwise choose the entering variable with $\qquad$ (3.3a)

largest reduced cost.

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*If we allow the possibility that several columns of A' are equal, we say the leftmost equal column dominates the others but not vice versa.

Delete all dominated columns from the tableau at
the beginning, then use any version of the simplex   (3.3b)
algorithm. (See [3,4] for algorithms for identifying
the set of dominated columns.)

The largest reduced cost rule for choosing entering variables does
not satisfy (3.2). Consider the program:*

			X_1	X_2	X_3	X_4
max	$5X_1 + 3X_2 + X_3 + X_4$					
subject to	$-X_1 + X_2 - 5X_3 - 3X_4 \geq -10$		0	0	0	0
	$-3X_1 - X_2 - X_3 - X_4 \geq -40$		10	0	0	0
	$+2X_1 - X_2 + 2.9X_3 + 3X_4 \geq -10$		12.5	2.5	0	0
			0	31 2/3	8 1/3	0

The column for X_4 dominates the column for X_3 but X_3 enters the basis
at the third iteration (see table at right).

If the matrix A' has U undominated columns then any version of the
simplex algorithm which satisfies (3.2) will use at most
$\binom{U+m}{m} \leq (U+m)^m$ pivot steps to find the optimum (or discover the LP is
unbounded).

To establish bounds on the expected number of pivots it suffices
to show that the size of U grows slowly with n, for m fixed.

Theorem 3.1: Fix m. There are constants $C_1 < e^{-m}$ and C_2 such
that, for n sufficiently large and A' generated so that (2.2) holds
the probability that there are at most

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*This example also shows that the "maximum increase" rule for
choosing entering variables does not satisfy (3.2). This corrects an
erroneous claim in the original manuscript.

$$C_2(Ln\ n)^{(m+1)Ln(m+1)\ +\ 1} \tag{3.4}$$

undominated columns is at least

$$1 - (Ln\ n)^{(m+1)Ln(m+1)} C_1^{Ln\ n} \tag{3.5}$$

We prove this result in section 4. From this we easily obtain

Theorem 3.2: Fix m. Let $P(n)$ be the expected number of pivots for an A' generated by (2.2). Then

$$\overline{\lim_{n\to\infty}}\ P(n)/(Ln\ n)^{m(m+1)Ln(m+1)\ +\ m} \le C_2^m \tag{36}$$

Proof: With probability $\ge$ (3.5) the number of pivots will be $\le (U+m)^m$, where U is given by (3.4). In the remaining cases, the worst number of pivots is n^m. The bound on C_1 implies that

$\lim_n (Ln\ n)^k C_1^{Ln\ n} n^m = 0$, for any k.

<div align="right">QED</div>

4. The Number of Undominated Columns in a Randomly Generated Matrix.

The assumption (2.2) has the effect of converting our problem to the following:

Suppose we generate an m by n matrix A in which each row is a permutation of {1, 2, ... n} chosen independently from the uniform distribution. Study the behavior of the random variable U = number of undominated columns.

This problem has been studied in [5,6], and we suspect there are earlier references. [5,6] show that the expected value of U is $O(\log n)^{m-1}$ for n large. However, we need different information to

obtain Theorem 3.2. The admittedly crude approach in this section uses Stirling's formula and elementary probability.

Lemma 4.1: Suppose we choose sets of size αn, βn independently from a set of n elements, $0 \leq \alpha$, $\beta \leq \frac{1}{2}$. There is $E < 1$ such that the probability that the intersection has exactly $\frac{1}{2} \alpha\beta n$ elements is $\leq E^{\alpha\beta n}$.

Proof. The probability in question is

$$\frac{\alpha n! \, \beta n! \, \alpha' b! \, \beta' n!}{n! \frac{1}{2}\alpha\beta n! \, \beta(1-\frac{1}{2}\alpha)n! \, \alpha(1-\frac{1}{2}\beta)n! \, (1-\beta-\alpha+\frac{1}{2}\alpha\beta)n!} \qquad (4.1)$$

where $\alpha + \alpha' = \beta + \beta' = 1$. If we apply Stirling's formula in the form $n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}} \leq n! \leq n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}} e^{1/12n}$ and the estimate $Ln(1+X) \leq X$, $X < 1$, we obtain the upper bound $RQ(2/e)^{\frac{1}{2}\alpha\beta n}$, where $Q = \exp(12n^{-1}(\alpha^{-1}+\beta^{-1}+\alpha'^{-1}+\beta'^{-1}))$ and $R \leq 8(\pi n\alpha\beta)^{-1}$. For $n\alpha\beta$ sufficiently large (≥ 10) we may take $E = (2/e)^{1/2}$.

QED

Lemma 4.2: There is $D < 1$ and Q such that, if $\alpha\beta n \geq Q$ the probability that the intersection as in lemma 4.1 has $\leq \frac{1}{2} \alpha\beta n$ elements is $\leq D^{\alpha\beta n}$.

Proof: If we denote by P_k the probability that the intersection has exactly k elements and look at the formula for P_k in factorials, we see that for $\alpha\beta n$ sufficiently large and $k \leq \frac{1}{2} \alpha\beta n$, $P_{k-1} \leq \frac{1}{2} P_k$.

Hence the probability is bounded by $\Sigma 2^{-i} E^{\alpha\beta n} = 2E^{\alpha\beta n}$. If $\alpha\beta n$ is suf-

ficiently large D can be chosen suitably.

<div align="right">QED</div>

We have abbreviated the proofs of lemmas 4.1 and 4.2 partly

because the details are cumbersome, and partly because these are known

properties of the hypergeometric distribution.

Lemma 4.3: There is a $B < 1$ and Q such that if $\alpha < \frac{1}{2}$ $\alpha^m n \geq 0$

and A is chosen so that (2.2) is satisfied, then the probability that

A has a column in which each entry is $\geq (1-\alpha)n$ is at least $1-B^{\alpha^m n}$.

Proof: The sets consisting of those columns whose first

(second) entry is among the αn largest in the first (second) row are

both of size αn and chosen independently. Hence the probability that

there are at least $\frac{1}{2}\alpha^2 n$ columns whose first two entries are

$\geq (1-\alpha)n$ is $\geq 1 - D^{\alpha^2 n}$, by lemma 4.2. Next we consider the set of

columns whose third entry is among the αn largest in the third row.

This set is independent of the set of columns whose first two entries

are among the αn largest. We may apply lemma 4.2 with $\beta = \frac{1}{2}\alpha^2$ to

conclude that the probability that there are at least $\frac{1}{4}\alpha^3 n$ columns

with entries $\geq (1-\alpha)n$ is $\geq (1-D^{\alpha^2 n})(1-D^{\frac{1}{2}\alpha^3 n})$. Continuing in this way

we conclude that the probability that there are at least $2^{-m+1}\alpha^m n$

columns, all of whose entries are among the αn largest is at least

$$(1-D^{\alpha^2 n})(1-D^{\frac{1}{2}\alpha^3 n})\ldots(1-D^{2^{-k+2}\alpha^k n}) \geq 1 - kD^{2^{-k+2}\alpha^k n} \qquad (4.3)$$

3 and Q may be chosen so that the conclusion follows.

<div align="right">QED</div>

the events corresponding to $P(j_1, \ldots j_q)$ occur for all $0 \leq q \leq L - 1$ and all sequences* $j_1, \ldots j_q$ all occur then the dominated columns must all be in $T(j_1, \ldots j_L)$, where the union is taken over all sequences of length L. The number of elements in the union and the lower bound on the probability of all the events occuring are as stated (note we do not assume the events are independent).

<div align="right">OED</div>

Theorem 4.6: For any Q, if A is randomly generated and satisfies (2.2) the probability that A has less than

$$Q(Ln\ n)^{mLn\ m+1}e \tag{4.6}$$

undominated columns is at least

$$1 - (Ln\ n)^{mLn\ m}B^{QLn\ n} \tag{4.7}$$

for n sufficiently large.

Proof: We apply lemma 4.5. Let $L = m\ Ln\ Ln\ n$ and $\alpha_1 = (QLn\ n/n)^{1/m}$. Let $\alpha_i^m = \alpha_i\alpha_{i+1}^m$. Then $m^L\alpha_1 \ldots \alpha_L n = (Ln\ n)^{mLn\ m}(QLn\ n)(n/QLn\ n)^\nu$, where $\nu = (1-1/m)^L \leq \exp(-L/m) = 1/Ln\ n$. Hence $n^\nu \leq e$. To justify (4.7) note that the expression (4.4) is in this case

$$1 - B^{QLn\ n}(\Sigma m^{q-1}) \geq 1 - B^{QLn\ n}(m^L).$$

<div align="right">QED</div>

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*The event corresponding to $q = 0$ is $\bigcap_1^n T(i)$ non-empty.

To obtain Theorem 3.1 from 4.6 replace m by m + 1 and choose Q so that $B^Q < e^{-m}$.

5. Concluding Remarks.

The expression (3.6) compares unfavorably with the expected value obtained in [1], which is essentially $(\text{Log } n)^{m^2+m}$. I conjecture that the analysis in section 4 can be improved to obtain this stronger result. It should also be mentioned that [1,2] also use the dominance idea (see the definition of the X-sets in section 5 of [1]). The contribution of this paper is to show that the key properties for this problem are (2.2), (3.2), and dominance, rather than geometric analysis.

These results extend to cases in which b is not necessarily feasible. To determine feasibility, one adds m columns corresponding to artificial variables. Since property (3.2) still holds, the extension is immediate.

Similarly, the possibility of a degenerate problem does not affect the analysis. The standard device of perturbing the right-hand side (lexicographic pivot rules) may be implemented in such a way that (3.2) still holds.

To replace (2.2) by (2.2)' we must show that the analysis in section 4 still works. For any matrix B let A be a matrix with all elements different such that $a_{ij} < a_{ik}$ if $b_{ij} > b_{ik}$. The number of undominated columns of B is $\leq$ the number of undominated columns of A. Further, if we permute row elements of B and A to obtain B*, A* # undominated B* $\leq$ # undominated A*. Thus the average number of undominated columns of B as we go through all possible permutations is

smaller than average number for A, to which the analysis in section 4 applies.

The results in this paper depend in a crucial way on $n \gg m$. It might be useful to look for dominance every few iterations, since new dominant columns may appear. Also, it might help to extend the concept of dominance to cases in which a non-negative multiple of one column dominates another.

Finally, we wish to mention work of Megiddo [7] on specialized algorithms for LPs with many variables and few constraints.

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